A NOVEL V-LINE RADON TRANSFORM AND ITS IMAGING APPLICATIONS

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ABSTRACT

The Radon transform (RT) on straight lines deals as mathematical foundation for many tomographic modalities (*e.g.* X-ray scanner, Positron Emission Tomography), using only primary radiation. In this paper, we consider a new RT defined on a pair of half-lines forming a letter V, arising from the modeling a two-dimensional emission imaging process by Compton scattered gamma rays. We establish its analytic inverse, which is shown to support the feasibility of the reconstruction of a *two-dimensional* image from scattered radiation collected on a *one-dimensional* collimated camera. Moreover, a filtered back-projection inversion method is also constructed. Its main advantages are algorithmic efficiency and computational rapidity. We present numerical simulations to illustrate the working. To sum up, the V-line RT leads not only to a *new imaging principle*, but also to a *new concept of detector* with high energetic resolution capable to collect the scattered radiation.

Index Terms— Radon transforms, image reconstruction, nuclear imaging, tomography.

1. INTRODUCTION

Collecting first order Compton scattered radiation by a two-dimensional gamma camera from an object-medium for three-dimensional imaging purposes [1, 2] has turned out to be a recent attractive alternative to conventional tomographic emission imaging, which uses only primary (or non-scattered) radiation. This new imaging principle is modeled by the so-called Conical Radon Transform (CRT) and has been supported by numerical simulations [2]. Later on, extensions of this idea have been proposed in various directions [3]. In this paper, we describe the implementation of this idea in the context of a one-dimensional gamma camera, which leads to a two-dimensional version of the CRT, which we call V-line Radon transform. The related imaging process may be realized, for example, on two-dimensional structures in material non-destructive testing as well as in biomedical imaging. Ideally, one can think of a radiation emitting flat object (or slice), in which Compton scattered radiation is collected by a collimated linear detector, in order to reconstruct the primary radiation source distribution of this object (see Figure 1).

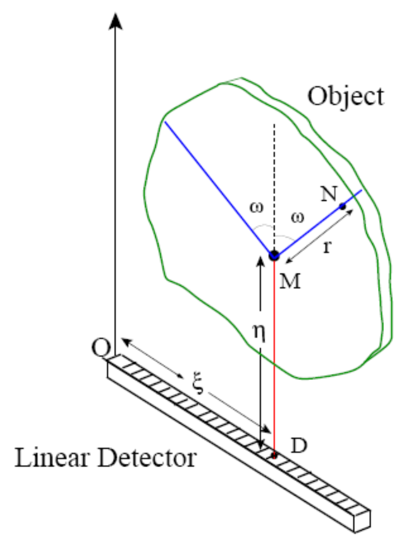

Fig. 1. Experimental setup and parameters used.

Section 2 shows how the image formation process by emission Compton scattered radiation is modeled and how the collected data by a linear collimated detector leads to a Radon transform on a pair of half-lines forming a letter V. This new integral transform, along with the conical Radon transform [1, 2, 3, 4], which we introduced a few years ago, becomes a new member of the rich family of Radon transforms [5], known so far in integral geometry as well as in tomographic imaging. Originally this V-line Radon transform has been proposed a decade ago by Basko *et.al.* [6] to model image formation in a two-dimensional Compton camera. However this Basko transform is in fact a V-line Radon transform with swinging axis around a detection site whereas

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the one considered here has a fixed axis direction. We give its properties, work out its kernel and its adjoint transform. In particular we establish its inverse under its analytical form and its corresponding filtered back-projection form. This last form has the advantage of reconstructing the image by fast algorithms. In section 3, we present numerical simulations on image reconstruction including a thyroid phantom to support the feasibility of this imaging process and present related comments. The paper ends with a short conclusion on the obtained results and opens some future research perspectives.

2. THE V-LINE RADON TRANSFORMATION

2.1. Image formation and the V-line Radon transform

Consider a 2D-object containing a non-uniform radioactivity source distribution, which is represented by a non-negative continuous function $f(x, y)$ with bounded support in $\{\mathbb{R}^2 | y > 0\}$. A collimated linear detector is set parallel to the plane of the object. It collects only outgoing radiation from the object which is parallel to the direction of the collimator holes (see Figure 1).

If the detector is set to absorb gamma photons at energies below the energy of primary photons emitted by the object, the photons have undergone a Compton scattering at a site $\mathbf{M}$ in the bulk of the object under a scattering angle ω. We assume that higher order scattering is neglected since it occurs with a much smaller probability.

The photon flux density at a detecting site $\mathbf{D}$ is due to the sum of the contribution of all emitting object point sources located on two half-lines starting at a site $\mathbf{M}$ and making an angle ω with the collimator axis direction, for all possible $\mathbf{M}$ along the axis of the collimator at $\mathbf{D}$.

Let $f(x, y)$ be an activity density function (object function), $\widehat{f}(\xi, \omega)$ the measured photon flux density at $\mathbf{D}$ under a scattering angle ω. Computing the photon flux density with the two-dimensional photometric law in the absence of radiation attenuation and for constant electronic density n_e, we can express $\widehat{f}(\xi, \omega)$ as

$$\widehat{f}(\xi, \omega) = K(\omega) \int_0^\infty \frac{d\eta}{\eta} T\mathbb{V}\, f(\xi, \eta, \omega), \qquad (1)$$

where $K(\omega) = n_e\, P(\omega)$, $P(\omega)$ differential Compton scattering cross-section (Klein-Nishina formula [7, 8]), and

$$T\mathbb{V}\, f(\xi, \eta, \omega) = \int_0^\infty \frac{dr}{r}$$
$$\left[f(\xi + r\sin\omega, \eta + r\cos\omega) + f(\xi - r\sin\omega, \eta + r\cos\omega) \right]. \qquad (2)$$

In the last integral, $f(x, y)$ is integrated on a discontinuous line having the form of a V-line with symmetry axis parallel to a fixed direction. Thus image formation by Compton scattered radiation in two dimensions leads to a new concept of Radon transform on a V-line.

2.2. Definition

We examine a simplified case of V-line Radon transform, for which $\eta = 0$. This transform in fact models the imaging process of a collimated one-dimensional Compton camera. Primary radiation emitted from the object bulk is scattered by a linear scattering detector, which lies along the Ox-axis of a cartesian coordinate system and collected later by a second absorbing detector along the vertical direction. This is of course an *ideal* hypothetic research camera, for which the V-line Radon transform describes the imaging process.

The $T\mathbb{V}$ transform of an activity density function $f(x, y)$, defined as the integral of this function along a V-line, each branch of which making an angle ω with the vertical direction, gives the detected photon flux density

$$T\mathbb{V} f(\xi, \omega) = \int_0^\infty \frac{dr}{r}$$
$$[f(\xi + r\sin\omega, r\cos\omega) \quad + f(\xi - r\sin\omega, r\cos\omega)], \qquad (3)$$

for $\xi \in \mathbb{R}$ and $0 \leq \omega < \pi/2$. For ease of notation we set $T\mathbb{V} f(\xi, \omega) = g(\xi, \omega)$ and observe that, because of the assumption on the support of f, the integral of eq. (3) is well-defined. ξ gives the position of the vertex on the Ox axis. The factor $1/r$ in the integrand accounts for the photometric law of photon propagation in two dimensions. From now on, we shall absorb $K(\omega)$ in the definition of f to keep the writing simple. Under the change of variables $t = \tan\omega$ and $z = r\cos\omega$, equation (3) reads

$$g(\xi, \omega) = G(\xi, t) = \int_0^\infty \frac{dz}{z} \left[f(\xi + tz, z) + f(\xi - tz, z) \right]. \qquad (4)$$

It is also useful to rewrite it as an integral transform, *i.e.*

$$g(\xi, \omega) = \int_{\mathbb{R}\times\mathbb{R}^+} dx\, dy\; k(x, y; \xi, \omega)\; f(x, y), \qquad (5)$$

with the kernel

$$k(x, y; \xi, \omega) = \frac{\cos\omega}{y}\, \delta(\cos\omega |x - \xi| - y\sin\omega).$$

2.3. The inverse transform $T\mathbb{V}^{-1}$

The inverse transform $T\mathbb{V}^{-1}$ can be worked out using Fourier transforms $\widetilde{f}(q, y)$ (resp. $\widetilde{g}(q, \omega)$) with respect to the variable x (resp. ξ) in $f(x, y)$ (resp. $g(\xi, \omega)$)), *i.e.*

$$g(\xi, \omega) = \int_{-\infty}^{\infty} dq\, \widetilde{g}(q, \omega)\, \exp(2i\pi q\xi), \qquad (6)$$

and

$$f(x, y) = \int_{-\infty}^{\infty} dq\, \widetilde{f}(q, y)\, \exp(2i\pi qx). \qquad (7)$$

Then equation (3) becomes

$$\widetilde{g}(q, \omega) = \int_0^\infty \frac{dr}{r}\, \widetilde{f}(q, r\cos\omega)\, 2\cos(2\pi qr\sin\omega). \qquad (8)$$

Upon change to variables z and t, and defining $\widetilde{G}(q,t) = \widetilde{g}(q,\omega)$ with $\widetilde{F}(q,z) = \widetilde{f}(q,z)/z$ one finds that

$$\widetilde{G}(q,t) = \int_0^\infty dz\ \widetilde{F}(q,z)\ 2\cos(2\pi qzt), \tag{9}$$

which is the cosine-Fourier transform. Thus we can write down the inverse formula immediately

$$\widetilde{F}(q,z) = 2|q| \int_0^\infty dt\ \cos(2\pi qtz)\ \widetilde{G}(q,t). \tag{10}$$

The inverse transform may be written as an integral transform with the inverse kernel

$$k^{-1}(x,z|\xi,t) = -\frac{z}{2\pi^2}\left[\frac{1}{(x-\xi+zt)^2} + \frac{1}{(x-\xi-zt)^2}\right]. \tag{11}$$

As it stands, this kernel is to be understood as a generalized function, or distribution and the corresponding integral should be taken as a Cauchy principal value.

2.4. Filtered back-projection inversion method (FBP-IM)

In this section we establish another formulation of the inversion procedure which lends itself more advantageous to algorithmic implementation. We call it *filtered back-projection* (FBP), due to its similarity to the one of standard Radon transform, but the novelty is that the FBP is carried on the V-lines but not on the straight lines. In the Radon transform the FBP is an exact inversion formula obtained by combining the action of the ramp filter and the back-projection operation of the Radon transform. In this section, we will demonstrate that the $T\mathbb{V}$ transform may be inverted essentially in the same way, the ramp filter and the back-projection operator associated to the $T\mathbb{V}$ operator playing an analogous role.

Technically the back-projection principle consists in assigning the value $g(\xi,\omega)$ to every point on the "projection" V-line, which has given rise to this value, and then to sum over all contributions for every V-line "projection". More precisely, we can say that the back-projection at angle ω in (x,y) is the sum of projections at angle ω at the points $\xi_1 = x + y\tan\omega$ and $\xi_2 = x - y\tan\omega$, where (x,y) is projected:

$$R_\omega(x,y) = g\left(x + y\tan\omega, \omega\right) + g\left(x - y\tan\omega, \omega\right). \tag{12}$$

The back-projection of every projection defines the back-projection operator $T\mathbb{V}^\#$ which is obtained by summing over every angle ω the expressions given in equation (12). Here a y-factor appears because of the measure dr/r in the definition of the projections (3).

Now the action of the ramp filter operator Λ over a function $f(x,y)$ in the first variable is defined in the Fourier domain by

$$\widetilde{\Lambda f}(q,y) = |q|\tilde{f}(q,y), \tag{13}$$

where the Fourier transform is taken on the first variable x. From equation (10) we have

$$f(x,y) = y\int_0^\infty dt\left[(\Lambda g)(x+ty,t) + (\Lambda g)(x-ty,t)\right]. \tag{14}$$

In terms of the angle ω, the inversion formula reads now

$$\frac{f(x,y)}{y} = \int_0^{\pi/2} \frac{d\omega}{\cos 2\omega}$$
$$[(\Lambda g)(x + y\ \tan\omega, \omega) + (\Lambda g)(x - y\ \tan\omega, \omega)]. \tag{15}$$

Setting $\mathcal{M}_\omega = g(\xi,\omega)/\cos 2\omega$ and knowing that

$$T\mathbb{V}^\# g(x,y) = \frac{1}{y}\int_0^{\pi/2} d\omega$$
$$[g\left(x + y\tan\omega, \omega\right) + g\left(x - y\tan\omega, \omega\right)], \tag{16}$$

we finally recover the original density $f(x,y)$ by a filtered-back projection as $f(x,y) = y2\,(T\mathbb{V}^\#\mathcal{M}_\omega\Lambda\, T\mathbb{V}f)(x,y)$.

3. NUMERICAL SIMULATIONS

We present now the results of numerical simulations. The original image (Fig. 2) of size 512×512 of length units is a thyroid phantom with small nodules. Fig. 3 shows the T$\mathbb{V}$ transform of a thyroid phantom with angular sampling rate $d\omega = 0.005$ rad and 314 projections ($\pi/2/0.005 = 314$) which are the images of Compton scattered radiation on the camera in terms of the distance ξ and the scattering angle ω. The reconstruction using FBP is given in Fig. 4. The artifacts are due to the limited number of projections. Moreover, back-projection on V-lines generates more artifacts than back-projection on straight lines, because of more spurious line intersections. A choice of a smaller $d\omega$ would improve image quality. Despite these limitations, the small structures in the object are clearly reconstructed. This result illustrates undoubtedly the feasibility of the new imaging modality, for which the main advantage resides in the use of a *one-dimensio- nal non-moving* Compton camera for *two-dimensional* image processing.

4. CONCLUSION

In this paper, a new class of Radon transform defined on a discontinuous line, having the shape of a V letter is presented. We construct its analytic inverse transform as well as the corresponding filtered back-projection inversion method. They allow two-dimensional image reconstruction from scattered radiation collected by a one-dimensional collimated camera. We have also performed numerical simulations to prove its practical viability. The obtained results provide a stimuli for tackling the case of the swinging V-line Radon transform, for a two-dimensional Compton camera imaging, as proposed by

[6]. Furthermore, the extension of this transform to a family of cones with swinging axis around a site in $\mathbb{R}^3$, for a concrete gamma camera without mechanical collimator, poses a real mathematical challenge to overcome in the future.

5. REFERENCES

[1] Nguyen M K and Truong T T, On an integral transform and its inverse in nuclear imaging *Inverse Problems* vol. 18, pp. 265-277, 2002.

[2] Nguyen M K, Truong T T, Bui H D and Delarbre J L, A novel inverse problem in γ-rays emission imaging *Inverse Problems in Science and in Engineering*, vol. 12, pp. 225-246, 2004.

[3] M. K. Nguyen, T. T. Truong, C. Driol and H. Zaidi, *On a novel approach to Compton scattered emission imaging*, IEEE Transactions in Nuclear Sciences, vol. 56, pp. 1430-1437, 2009.

[4] T. T. Truong, M. K. Nguyen and H. Zaidi, The mathematical foundations of 3D Compton scatter emission imaging, *International Journal of Biomedical Imaging, Special Issue on Mathematics in Biomedical Imaging*, doi: 10. 1155/2007/92780, 2007.

[5] Radon J, Über die Bestimmung von Funktionnen durch ihre Integralwerte längs gewisser Mannigfaltikeiten *Ber.Verh.Sachs.Akad.Wiss. Leipzig-Math.-Natur.Kl.*, vol. 69, pp. 262-277, 1917.

[6] R. Basko, G.L. Zeng, G. T. Gullberg, Analytical reconstruction formula for the one-dimensional Compton camera, *IEEE Trans. Nucl. Sci.*, vol.44, pp. 1342-1346, 1997.

[7] R D Evans, The atomic nucleus. New York, McGraw-Hill, 1955.

[8] Barrett H H, The Radon Transform and its Applications in *Progress in Optics*, vol. 21, pp. 219-286, Ed. E. Wolf, North Holland, 1984.

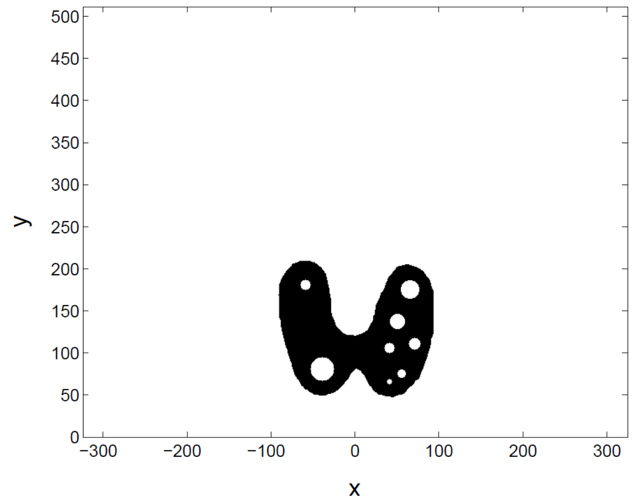

Fig. 2. Original thyroid phantom.

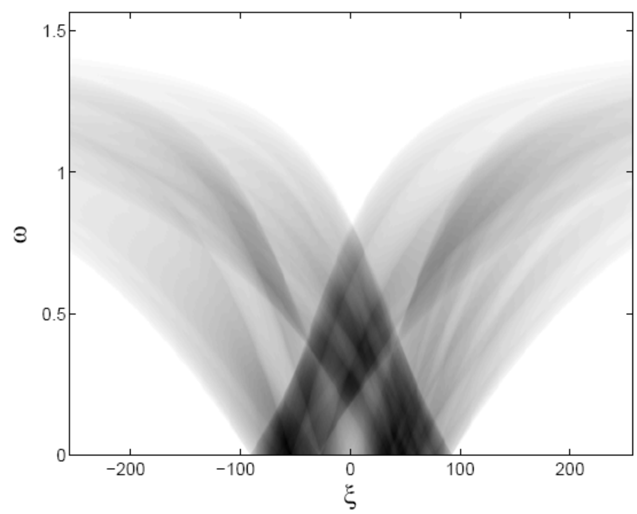

Fig. 3. The $T\mathbb{V}$ transform of the thyroid image shown in Figure 2 with $d\omega = 0.005$ rad.

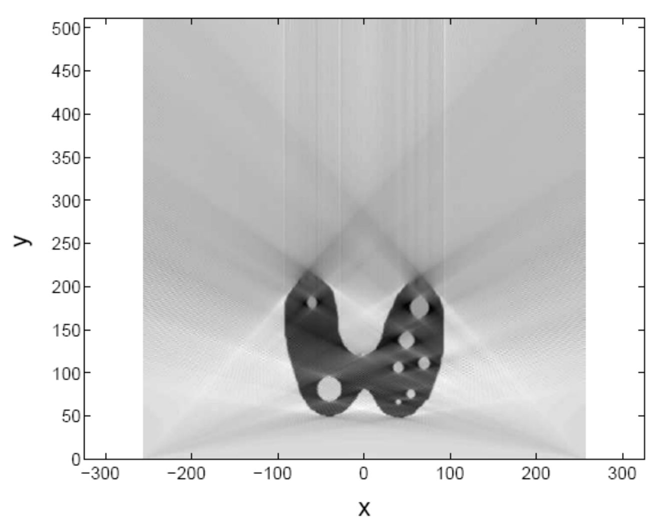

Fig. 4. FBP-IM reconstruction ($d\omega = 0.005$ rad).